B.Sc. (Maths) - part II
paper - VI

Topic: Normalizer (Group Theory)

Normalizer of an element of a group

Def:- The set of all those elements of a group G which commute with a fixed element of group G is called normalizer of an element a of the group and denoted by $N(a)$.

Hence $N(a) = \{x \in G : ax = xa\}$

Theorem ① The normalizer $N(a)$ of an element $a \in G$ is sub group of G.

proof:- We have $N(a) = \{x \in G : ax = xa\}$

Let $x_1, x_2 \in N(a)$ where $x_1, x_2 \in G$ and $a \in G$

In order to show that $N(a)$ is subgroup of G we need to show that $x_1 x_2^{-1} \in N(a)$

Since $x_1, x_2 \in N(a)$ we have

$$ax_1 = x_1 a \text{ and } ax_2 = x_2 a$$

we have $ax_2 = x_2 a \Rightarrow$

$$x_2^{-1}(ax_2)x_2^{-1} = x_2^{-1}(x_2 a)x_2^{-1}$$

$\Rightarrow x_2^{-1}a(x_2x_2^{-1}) = (x_2^{-1}x_2)ax_2^{-1}$

$\Rightarrow x_2^{-1}a = ax_2^{-1} \Rightarrow x_2^{-1} \in N(a)$

Now we shall show that

$x_1x_2^{-1} \in N(a)$

we have $a(x_1x_2^{-1}) = (ax_1)x_2^{-1}$

$= (x_1a)x_2^{-1} \quad \because ax_1 = x_1a$

$= x_1(ax_2^{-1})$

$= x_1(x_2^{-1}a) \quad \because x_2^{-1} \in N(a)$

and hence $(ax_2^{-1}) = (x_2^{-1}a)$

$= (x_1x_2^{-1})a$

$\therefore x_1x_2^{-1} \in N(a)$

Thus $x_1, x_2 \in N(a) \Rightarrow x_1x_2^{-1} \in N(a)$

Therefor $N(a)$ is a sub group of G.